The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Final Exam ***Solutions 1.0***

Date: December 12, 2025 Course: ECE 313 Evans

Name: **Solutions**
Last, First

- **Exam duration**. The exam is scheduled to last two hours.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **No AI tools allowed**. As mentioned on the course syllabus, you may not use GPT or other AI tools during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

Problem	Point Value	Your Score	Topic
1	18		Continuous-Time System Properties
2	18		Discrete-Time Convolution
3	16		Continuous-Time Integrators
4	14		Discrete-Time Filter Design
5	16		Continuous-Time Sinusoidal Amplitude Modulation
6	18		Discrete-Time Mystery Systems
Total	100		

Problem 1. Continuous-Time System Properties. *18 points*

Each continuous-time system has input $x(t)$ and output $y(t)$, and $x(t)$ and $y(t)$ might be complex-valued.

Determine if each system is linear or nonlinear, and time-invariant or time-varying.

You must either prove that the system property holds in the case of linearity or time-invariance, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Part	*System Name*	*System Formula*	*Linear?*	*Time-Invariant?*
(a)	Finite Impulse Response (FIR) filter	$y(t) = x(t) + x(t-1)$ for $-\infty < t < \infty$	**YES**	**YES**
(b)	Inverse	$y(t) = \frac{1}{x(t)}$ for $-\infty < t < \infty$	**NO**	**YES**
(c)	Differentiator	$y(t) = \frac{d}{dt} x(t)$ for $-\infty < t < \infty$	**YES**	**YES**

***Linearity.* We'll first apply the all-zero input test which is to input $x(t) = 0$ for all time t under observation and if the output $y(t)$ is not zero for all time under observation, then the system is not linear. Otherwise, we'll have to apply the definitions for homogeneity and additivity. All-zero input test is a special case of homogeneity $a\,x(t) \rightarrow a\,y(t)$ when the constant $a = 0$.**

***Time-Invariance*: If the current output value $y(t)$ depends only on current input $x(t)$ and not on any other input/output values, it is pointwise operation. Pointwise operations are time-invariant.**

(a) Finite impulse response (FIR) filter: $y(t) = x(t) + x(t-1)$ for $-\infty < t < \infty$. *6 points.*

***Linearity*: Passes all-zero input test. Need to check the following properties:**

- ***Homogeneity:* Input $a\,x(t)$. $y_{scaled}(t) = (a\,x(t)) + (a\,x(t-1)) = a\,y(t)$. YES.**
- ***Additivity*: Input $x_1(t) + x_2(t)$. $y_{additive}(t) = (x_1(t) + x_2(t)) + (x_1(t-1) + x_2(t-1)) = (x_1(t) + x_1(t-1)) + (x_2(t) + +x_2(t-1)) = y_1(t) + y_2(t)$. YES.**

***Time-Invariance*: Input $x(t-t_0)$. $y_{shifted}(t) = x(t-t_0) + x(t-t_0-1) = y(t-t_0)$. YES.**

(b) Inverse: $y(t) = \frac{1}{x(t)}$ for $-\infty < t < \infty$. *6 points.*

***Linearity*: Fall all-zero input test.** When **$x(t) = 0, y(t) = \frac{1}{0}$ which is not 0.**

***Time-Invariance*: All pointwise operations are time-invariant.**

(c) Differentiator: $y(t) = \frac{d}{dt} x(t)$ for $-\infty < t < \infty$. *6 points.*

***Linearity*: Passes all-zero input test. Need to check the following properties:**

- ***Homogeneity:* Input $a\,x(t)$. $y_{scaled}(t)) = \frac{d}{dt}\left(a\,x(t)\right) = a\,\frac{d}{dt}\,x(t) = a\,y(t)$. YES.**
- ***Additivity*: Input $x_1(t) + x_2(t)$. $y_{additive}(t) = \frac{d}{dt}\left(x_1(t) + x_2(t)\right) = \frac{d}{dt}\,x_1(t) + \frac{d}{dt} x_2(t) =) = y_1(t) + y_2(t)$. YES.**

***Time-Invariance*: All pointwise operations are time-invariant. See above. YES.**

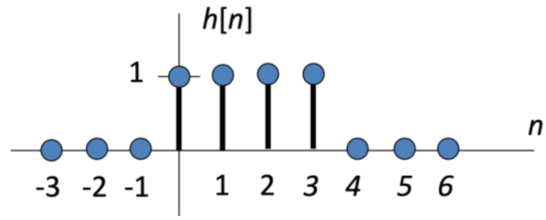
Problem 2. Discrete-Time Convolution. *18 points*

Consider a discrete-time linear time-invariant (LTI) system with impulse response plotted on the right of

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

The output of the LTI system is the convolution of the impulse response $h[n]$ and the input signal $x[n]$:

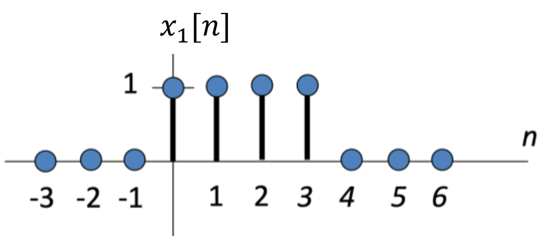
$$y[n] = h[n] * x[n]$$



***Note*: This is an impulse response of a four-point averaging filter.**

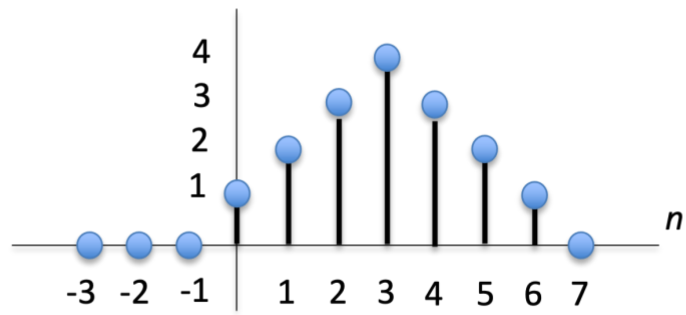
(a) Let the input signal be $x_1[n] = h[n]$.

Here, $x_1[n]$ has four non-zero values.



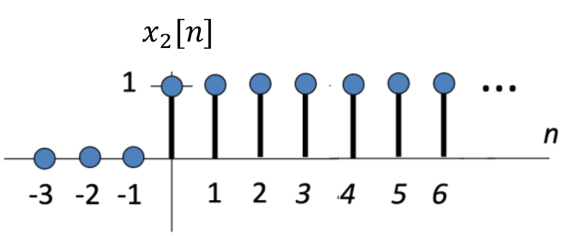
i. Give a formula for $y[n]$. *3 points*.

ii. Plot $y[n]$. *3 points*.



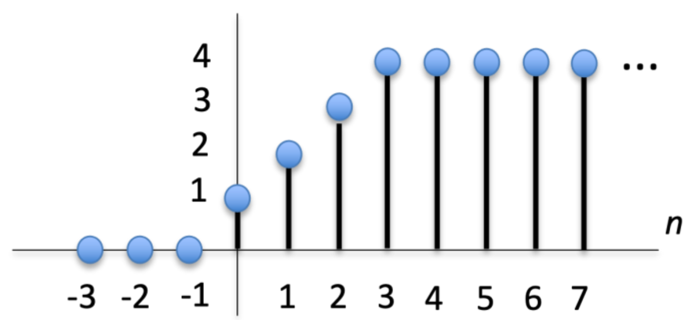
$$\boldsymbol{y[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2] + 4\,\delta[n-3] + 3\,\delta[n-4] + 2\,\delta[n-5] + \delta[n-6]}$$

(b) Let the input signal be $x_2[n] = u[n]$ which is the unit step function.



i. Give a formula for $y[n]$. *3 points*.

ii. Plot $y[n]$. *3 points*.



$$\boldsymbol{y[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2] + 4\,u[n-3]}$$

(c) Let the input signal be

$x_3[n] = \cos(\hat{\omega}_0\, n)$ for $-\infty < n < \infty$.

Give all possible values for $\hat{\omega}_0$ for which $y[n] = 0$ for $-\infty < n < \infty$. *6 points.*

Easier to work in the frequency domain.

$h[n]$ is the impulse response of a four-point averaging filter. **See the handout [Designing Averaging Filters](#).**

$$\boldsymbol{H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n} = \sum_{m=0}^{3} h[n]\, z^{-n}}$$

$\boldsymbol{H(z) = 1 + z^{-1} + z^{-2} + z^{-3}}$ **for** $\boldsymbol{z \neq 0}$

$H(z)$ has zeros on unit circle at discrete-time frequencies (angles) of $\hat{\omega}_0 \in \{\frac{\pi}{2}, \pi, \frac{3}{2}\pi\}$.

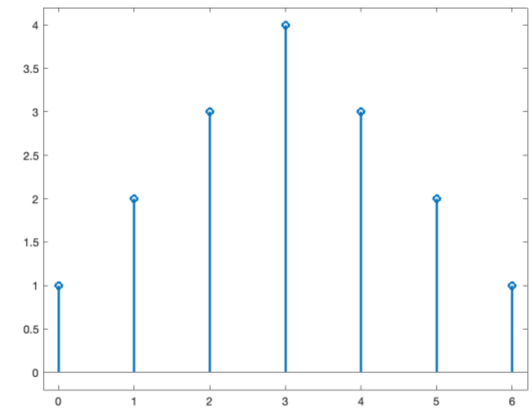
These frequencies are zeroed out by the filter.

Since the discrete-time frequency domain is periodic with periodicity 2π, the complete of values for $\hat{\omega}_0 \in \left\{\frac{\pi}{2}, \pi, \frac{3}{2}\pi\right\} + 2\pi k$ for integer k.

Solutions for using Matlab

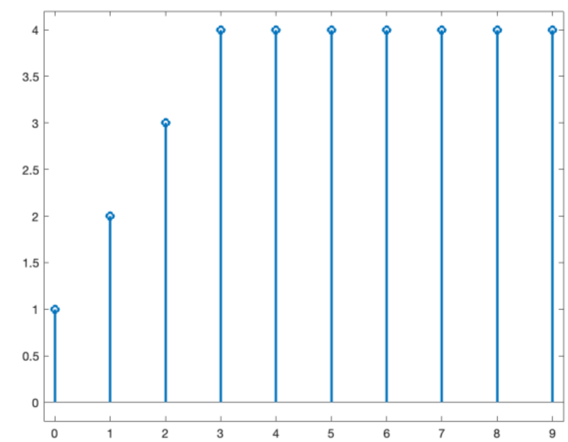
```matlab
% Part (a)
h = [1 1 1 1];
x1 = [1 1 1 1];
y = conv(h, x1);
n = 0 : 6;
stem(n, y, 'LineWidth', 2);
xlim( [-0.2 6.2] );
ylim( [-0.2 4.2] );

% y = [ 1 2 3 4 3 2 1 ]
```



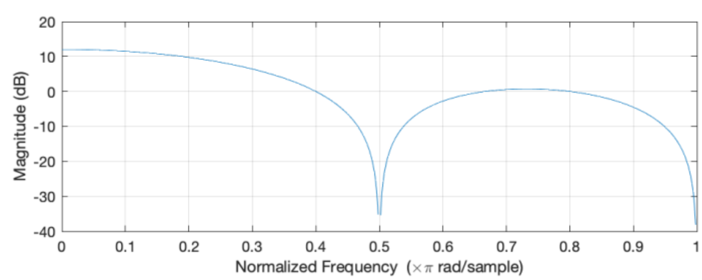
```matlab
% Part (b)
% conv convolves two finite
% length sequences
% using filter is better here
h = [1 1 1 1];
N = 10;
x2 = ones(1, N);
y = filter(h, 1, x2);
n = 0 : N-1;
stem(n, y, 'LineWidth', 2);
xlim( [-0.2 N-1+0.2] );
ylim( [-0.2 4.2] );

% y = [ 1 2 3 4 4 4 4 … ]
```



```matlab
% Part (c)
% Plot the frequency response
% of the FIR filter h[n]
h = [1 1 1 1];
freqz(h);

% Magnitude response is zero at
% frequencies pi/2, pi, -pi/2.
% Due to periodicity of 2pi in
% DT freq. domain, -pi/2 = 3pi/2
```



(c) Alternate solution in the time-domain

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x_3[n-m] = \sum_{m=0}^{3} h[m]\, x_3[n-m]$$

$$y[n] = x_3[n] + x_3[n-1] + x_3[n-2] + x_3[n-3]$$

$$x_3[n] = \cos(\hat{\omega}_0\, n)$$

We need to find $\hat{\omega}_0$ so that any four consecutive samples of $x_3[n]$ add to 0:

1, -1, 1, -1, … for $\hat{\omega}_0 = \pi$

1, 0, -1, 0, … for $\hat{\omega}_0 = \frac{1}{2}\pi$ or $\hat{\omega}_0 = -\frac{1}{2}\pi$

Problem 3. Continuous-Time Integrator. *16 points*

The integrator is a building block in continuous-time systems.

Denote the input signal as $x(t)$ and the output signal as $y(t)$.

(a) When observing the continuous-time integrator for $-\infty < t < \infty$,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

The system is linear and time-invariant (LTI).

i. Give a formula for the impulse response. 3 *points*.

Let $\boldsymbol{x(t) = \delta(t)}$. $\boldsymbol{y(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)}$ **where** $\boldsymbol{u(t) = \begin{bmatrix} 1 & \text{for } t > 0 \\ ? & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{bmatrix}}$

ii. Give a formula for the step response. *3 points*

Let $\boldsymbol{x(t) = u(t)}$. **So,** $\boldsymbol{y(t) = h(t) * x(t) = u(t) * u(t) = t\, u(t)}$

$$\boldsymbol{u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} d\tau = t \;\; \text{for } t > 0}$$

due to $\boldsymbol{u(\tau) = 1}$ **for** $\boldsymbol{\tau > 0, u(t-\tau) = 1}$ **for** $\boldsymbol{t > \tau}$ **and** $\boldsymbol{u(\tau)\, u(t-\tau) = 1}$ **for** $\boldsymbol{0 < \tau < t}$

iii. Give a formula for the frequency response. *3 points*

$$\boldsymbol{H(j\,\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\,\omega}\, dt = \int_{-\infty}^{\infty} u(t)\, e^{-j\,\omega\,t}\, dt = \int_{0}^{\infty} e^{-j\,\omega\,t}\, dt = \left.\frac{e^{-j\,\omega\,t}}{-j\,\omega}\right]_0^{\infty}}$$

$$\boldsymbol{H(j\,\omega) = \left(\lim_{t\to\infty} \frac{e^{-j\,\omega\,t}}{-j\,\omega}\right) + \frac{1}{j\,\omega} = \pi\,\delta(\omega) + \frac{1}{j\,\omega}}$$

As $\boldsymbol{t \to \infty}$**, the expression** $\boldsymbol{e^{-j\,\omega\,t}}$ **oscillates. We'll look up the answer in a table.**

iv. Excluding the response to zero frequency, would you describe the frequency response as lowpass, highpass, bandpass, bandstop, or allpass? Why? *3 points*

Lowpass. Excluding $\boldsymbol{\omega = 0}$**, magnitude response 1/|ω| decays with increasing frequency.**

(b) When observing the continuous-time integrator for $t \geq 0$,

$$y(t) = C_0 + \int_{0}^{t} x(\tau)\, d\tau$$

where C_0 is a real-valued constant.

i. What is the initial condition(s)? *2 points*.

$$\boldsymbol{y(t) = C_0 + \int_{0}^{0} x(\tau)\, d\tau = C_0}$$

ii. What value(s) should the initial condition(s) have as a necessary condition for LTI properties to hold? *2 points.*

$\boldsymbol{C_0 = 0}$

Problem 4. Discrete-Time Filter Design. *14 points*.

This problem asks you to design a discrete-time linear time-invariant (LTI) invariant filter to remove specific frequencies from a signal.

Harmonics can occur due to nonlinear distortion in a system. Design a discrete-time finite impulse response (FIR) filter to remove harmonics of continuous-time frequency, f_0. Harmonic frequencies of f_0 are $f_0, 2f_0, 3f_0, \ldots$ and $-f_0, -2f_0, -3f_0, \ldots$

(a) For sampling rate f_s, how many harmonics in positive frequencies, *N*, would be captured by sampling? *5 points*.

From the Sampling Theorem, $f_s > 2\, f_{max}$.

With $f_{max} = N\, f_0$, $f_s > 2\,(N\, f_0)$ and hence $N < \frac{f_s}{2\, f_0}$.

Since N is an integer, $N = \text{floor}\left(\frac{f_s}{2\, f_0}\right)$

(b) Give formulas for the zeros of the LTI transfer function in the *z*-domain. *5 points*.

A zero will be placed on the unit circle (radius 1) at angle equal to the discrete-time frequency $\hat{\omega}_k$ being eliminated and its negated value where

$$\hat{\omega}_k = 2\pi \frac{f_o}{f_s}\, k$$

and the zero locations would be at

$$z = e^{j\,\hat{\omega}_k} \text{ and } z = e^{-j\,\hat{\omega}_k}$$

for $k = 1, 2, \ldots, N$. So, there are $2N$ zeros.

(c) Give the discrete-time input-output relationship for the FIR filter assuming the input signal is $x[n]$ and the output signal is $y[n]$. *4 points*

For each pair of zeros, we have a second-order FIR nulling filter to remove frequency $\hat{\omega}_k$:

$$H(z) = \left(1 - e^{j\,\hat{\omega}_k}\, z^{-1}\right)\left(1 - e^{-j\,\hat{\omega}_k}\, z^{-1}\right) = 1 - (2\,\cos\hat{\omega}_k)\, z^{-1} + z^{-2}$$

which has the input-output relationship

$$y[n] = x[n] - (2\,\cos\hat{\omega}_k)\, x[n-1] + x[n-2]$$

We would cascade N second-order FIR nulling filters.

In the special case that $f_s = 2\, N\, f_0$, the angles of the $2N$ zeros are

$$\hat{\omega}_k = 2\pi \frac{f_o}{2\, N\, f_0}\, k = \frac{\pi}{N} k \;\; for\; k = 1, 2, \ldots, 2\, N$$

Which means the $2N$ zeros are uniformly spaced around the unit circle starting at $z = 1$.

This corresponds to $2N$ roots of unity. This is known an FIR comb filter.

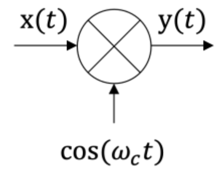
The input-output relationship is

$$y[n] = x[n] - x[n - 2N]$$

Problem 5. Continuous-Time Sinusoidal Amplitude Modulation. *16 points*.

Continuous-time sinusoidal amplitude modulation multiplies the input signal $x(t)$ by a sinusoidal signal of fixed frequency ω_c in rad/s to give the output signal $y(t)$ where

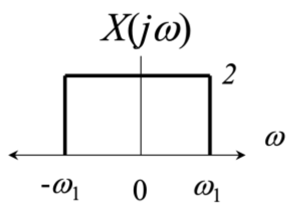
$$y(t) = x(t)\ \cos(\omega_c\ t)$$

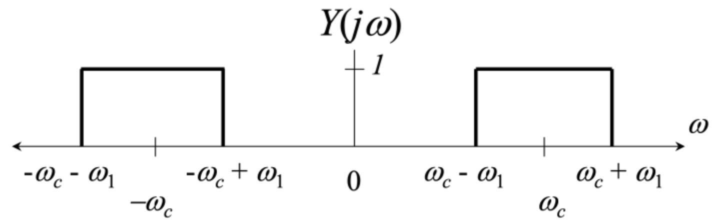


By taking the Fourier transform of both sides, we obtain the Modulation Property:

$$Y(j\ \omega) = \frac{1}{2}\ X(j\ (\omega + \omega_c)) + \frac{1}{2}\ X(j\ (\omega - \omega_c))$$

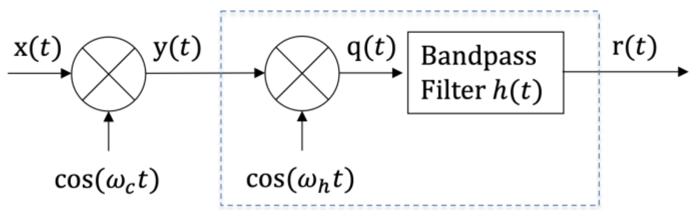
The term $\frac{1}{2}\ X(j\ (\omega + \omega_c))$ shifts the frequency content of $X(j\ \omega)$ left in frequency by ω_c and scales the amplitude by ½ and the term $\frac{1}{2}\ X(j\ (\omega - \omega_c))$ shifts the frequency content of $X(j\ \omega)$ right in frequency by ω_c and scales the amplitude by ½. Here's an example using an ideal lowpass spectrum for $X(j\ \omega)$:



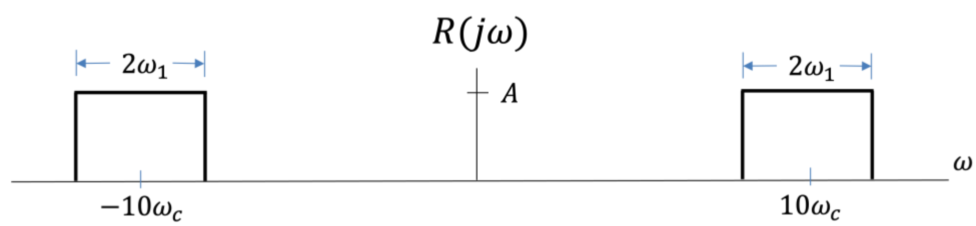


Note $\omega_c > \omega_1$. Please use the above Fourier transforms for $x(t)$ and $y(t)$ throughout this problem.

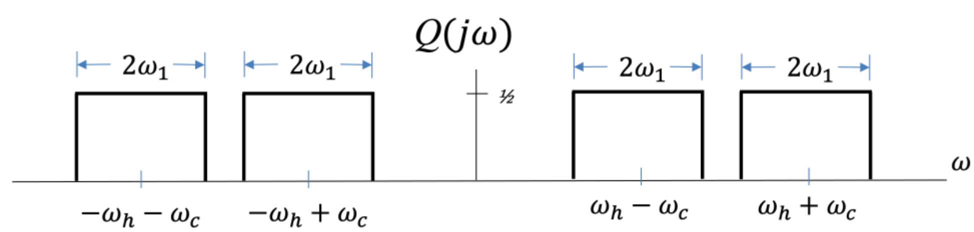
We can extend this idea to add a second stage of sinusoidal amplitude modulation (in the dashed box) to increase the carrier frequency ω_c to an even higher carrier frequency as shown on the right:



To achieve the frequency content of $r(t)$ shown on the right, complete the following:



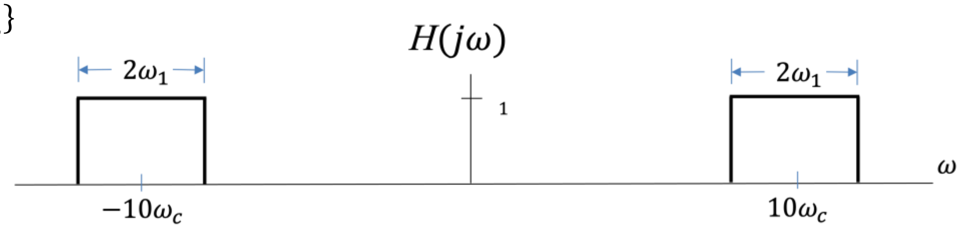
(a) Draw the frequency response of $q(t)$. *4 points*



(b) What is the value of ω_h in terms of the other parameters in the system? *4 points*.

$\boldsymbol{\omega_h \in \{\ 9\ \omega_c,\ 11\ \omega_c,\ -9\ \omega_c,\ -11\ \omega_c,\}}$

(c) Draw the frequency response of the bandpass filter $H(j\ \omega)$. *4 points*.

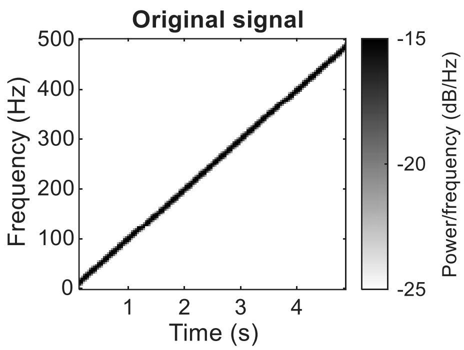


(d) What is the value of A? *4 points*.

$$\boldsymbol{A = \frac{1}{2}}$$

This approach, which uses two stages for sinusoidal amplitude modulation, is called *heterodyning*. For example, the first stage could be in analog circuits and the second stage in RF circuits. The first stage, once implemented, can be reused in many designs.

Problem 6. Discrete-Time Mystery Systems. *18 points*.



Consider a discrete-time chirp signal $x[n]$ obtained by sampling a chirp signal that sweeps from 0 to 500 Hz.

$$x(t) = \cos(2\pi f_1 t + 2\pi \mu t^2)$$

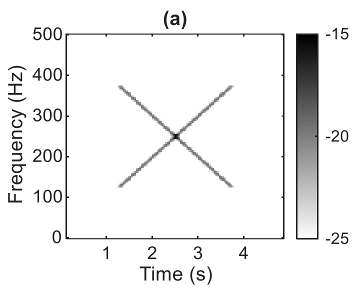
Where $f_1 = 0$, $f_2 = 500$ Hz, $\mu = \frac{f_2 - f_1}{2t_{\max}} = \frac{500\text{ Hz}}{10\text{s}}$, and $f_s = 1000$ Hz

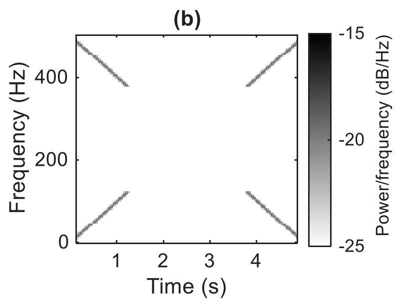
Assuming $x[n]$ is the input (original signal shown), indicate which of the systems (1-8) would produce each output (a), (b) and (c). Justify your answer.

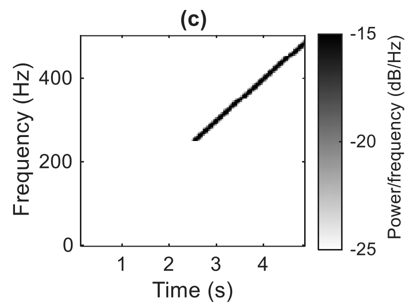
Here, $\downarrow_2$ means downsampling by a factor of two and $\uparrow_2$ means upsampling by a factor of two.

$$\downarrow_2 \{x[n]\} = x[2n] \qquad \uparrow_2 \{x[n]\} = \begin{cases} x[m]|_{m=\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} = \left(\frac{1}{2} + \frac{1}{2}\cos(\pi n)\right) x[m]|_{m=\frac{n}{2}}$$

Hint: Draw the spectrograms corresponding to $\downarrow_2 \{x[n]\}$ and $\uparrow_2 \{x[n]\}$.





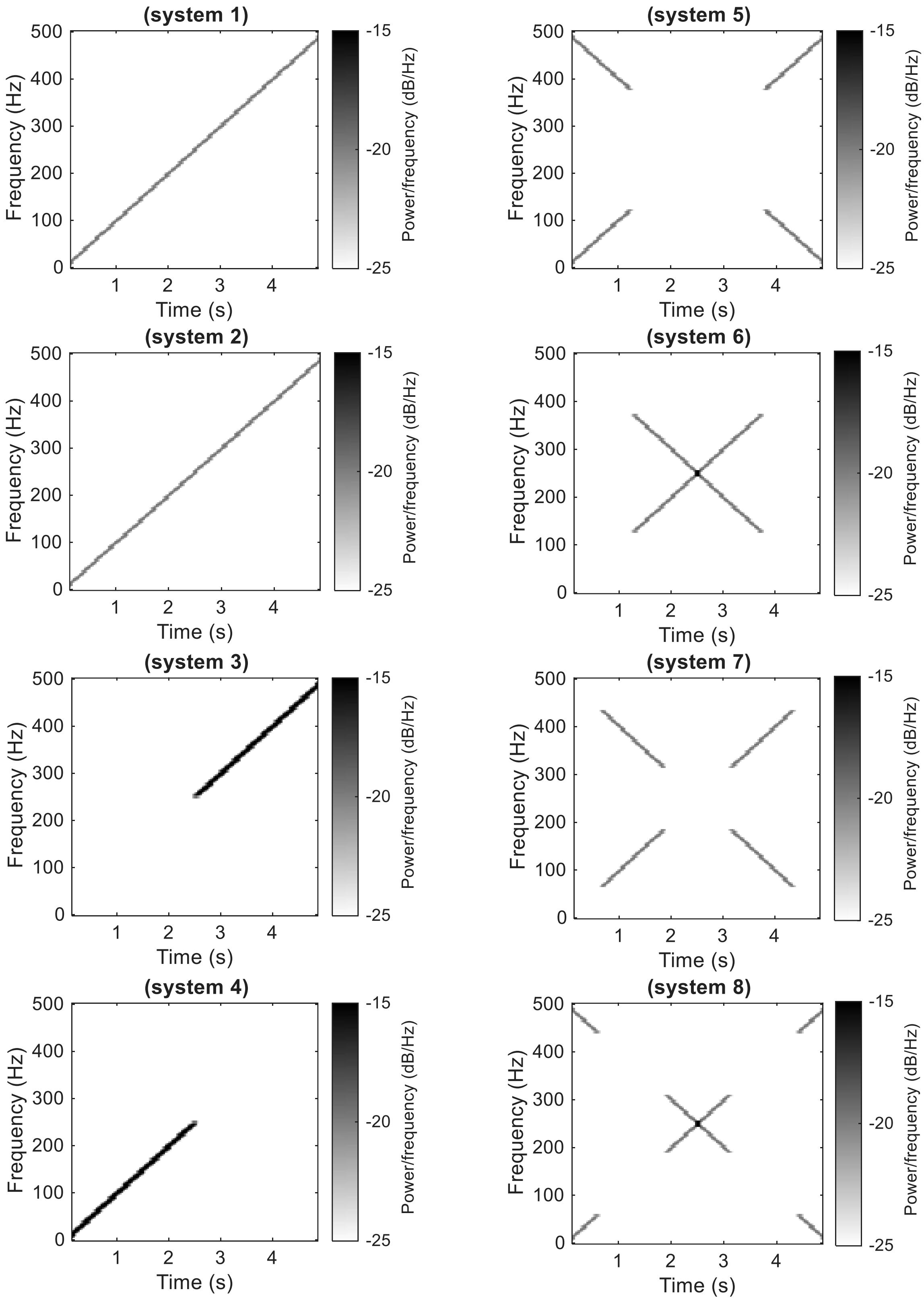


(1)	$x[n] \rightarrow$ $\uparrow_2$ $\rightarrow$ Lowpass $\rightarrow$ $\downarrow_2$ $\rightarrow y_1[n]$
(2)	$x[n] \rightarrow$ $\uparrow_2$ $\rightarrow$ Highpass $\rightarrow$ $\downarrow_2$ $\rightarrow y_2[n]$
(3)	$x[n] \rightarrow$ $\uparrow_2$ $\rightarrow$ Bandpass $\rightarrow$ $\downarrow_2$ $\rightarrow y_3[n]$
(4)	$x[n] \rightarrow$ $\uparrow_2$ $\rightarrow$ Bandstop $\rightarrow$ $\downarrow_2$ $\rightarrow y_4[n]$
(5)	$x[n] \rightarrow$ $\downarrow_2$ $\rightarrow$ Lowpass $\rightarrow$ $\uparrow_2$ $\rightarrow y_5[n]$
(6)	$x[n] \rightarrow$ $\downarrow_2$ $\rightarrow$ Highpass $\rightarrow$ $\uparrow_2$ $\rightarrow y_6[n]$
(7)	$x[n] \rightarrow$ $\downarrow_2$ $\rightarrow$ Bandpass $\rightarrow$ $\uparrow_2$ $\rightarrow y_7[n]$
(8)	$x[n] \rightarrow$ $\downarrow_2$ $\rightarrow$ Bandstop $\rightarrow$ $\uparrow_2$ $\rightarrow y_8[n]$

(a) Exact match to system (6).
Partial match to system (8).

(b) Exact match to system (5).
Partial match to system (7).

(c) Exact match to system (3)


(system 1)
(system 2)
(system 3)
(system 4)
(system 5)
(system 6)
(system 7)
(system 8)
Frequency (Hz)
Time (s)
Power/frequency (dB/Hz)
0
100
200
300
400
500
1
2
3
4
-15
-20
-25

```matlab
%% MATLAB code to generate spectrograms for problem 6
%% by Mr. Dan Jacobellis, UT Austin
blockSize = 256; overlap = 255;
fs = 1000; Ts = 1 / fs;
tmax = 5; t = 0 : Ts : tmax;
%% Input chirp signal
f1 = 0; f2 = fs/2;
mu = (f2 - f1) / (2*tmax);
x = cos(2*pi*f1*t + 2*pi*mu*(t.^2));
figure; spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15])
title('Original signal')

%% apply systems
N = 600; win = gausswin(N+1, 2.5);
lowpass_coeff = firhalfband(N, win, 'low');
highpass_coeff = firhalfband(N, win, 'high');
bandpass_coeff = fir1(N, [1/4,3/4], win, 'bandpass');
bandstop_coeff = fir1(N, [1/4,3/4], win, 'stop');
up2 = @(x) upsample(x,2);
down2 = @(x) downsample(x,2);
lowpass = @(x) conv(x,lowpass_coeff,'same');
highpass = @(x) conv(x,highpass_coeff,'same');
bandpass = @(x) conv(x,bandpass_coeff,'same');
bandstop = @(x) conv(x,bandstop_coeff,'same');
s1 = down2(lowpass(up2(x)));
figure; spectrogram(s1, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 1)')
s2 = down2(highpass(up2(x)));
figure; spectrogram(s2, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 2)')
s3 = down2(bandpass(up2(x)));
figure; spectrogram(s3, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 3)')
s4 = down2(bandstop(up2(x)));
figure; spectrogram(s4, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 4)')
s5 = up2(lowpass(down2(x)));
figure; spectrogram(s5, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 5)')
s6 = up2(highpass(down2(x)));
figure; spectrogram(s6, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 6)')
s7 = up2(bandpass(down2(x)));
figure; spectrogram(s7, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 7)')
s8 = up2(bandstop(down2(x)));
figure; spectrogram(s8, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray; colormap(flipud(colormap)); clim([-25,-15]); title('(system 8)')
```